Wave Propagation in Sinusoidally Stratified Dielectric Media

T. TAMIR, SENIOR MEMBER, IEEE, H. C. WANG, AND A. A. OLINER, FELLOW, IEEE

Summary—**The dispersion properties and the fields of electromagnetic waves are investigated for propagation in a stratified infinite medium. The stratification is characterized by a dielectric constant which, along one coordinate, is modulated sinusoidally about an average value. A systematic and comprehensive study is presented for the case of *H* modes for which the pertinent wave equation is in the form of a Mathieu differential equation. The modes and dispersion characteristics are analyzed in terms of a "stability" chart, which is customary in the study of the Mathieu equation. Results are obtained for an unbounded medium and for a waveguide filled with the modulated medium. Also, the reflection occurring at an interface between free space and a semi-infinite medium of this type is examined. In addition to these rigorous results for arbitrary values of modulation, simple analytical expressions are given for all of these cases where the modulation in the dielectric is small. It is shown that the fields are then expressible in terms of the fundamental and the two nearest space harmonics. The fields within a unit cell in the stratified medium are calculated for both small and large modulation and for frequencies up through the second pass band. It is of interest that the variation of the fields is not, in general, simply related to the variation of the dielectric constant within a cell.**

I. INTRODUCTION

THE STUDY described herein deals with the propagation of electromagnetic waves in a medium possessing a dielectric constant which is sinusoidally stratified along one coordinate. The interest in this specific problem was stimulated by several topics to which the present work either has direct application or serves as a basis for further developments.

A first area of application is concerned with electromagnetic wave propagation through a compressible medium which is influenced by acoustic or other mechanical waves. As an example, the results are applicable to acoustically-modulated plasma media in the range of frequencies above the plasma frequency ω_p. A second area of application regards the stratified medium as a first step in the analysis of a sinusoidally-modulated dielectric slab antenna which employs a layer of the above mentioned medium. Aside from the motivations and possible areas of application, the case investigated is a canonical one in the theory of wave propagation in periodic structures. The systematic and comprehensive treatment contained herein of both the propagation characteristics and the field distributions should therefore be of value in itself.

Manuscript received November 4, 1963; revised February 20, 1964. The investigation described herein was sponsored concurrently by the U. S. Air Force Cambridge Research Laboratories, Bedford, Mass. under Contract No. AF-19(604)-7499, and by the Office of Naval Research, Washington D. C. under Contract No. Nonr-839(28).
The authors are with the Electrophysics Department, Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

The propagation of waves in periodically-stratified media was discussed as early as 1887 by Lord Rayleigh,[1] who recognized that this electromagnetic problem was characterized by the Hill and Mathieu differential equations and who analyzed some of the properties of these equations. It should also be mentioned that, both before and after Lord Rayleigh, the Hill and Mathieu equations were investigated in connection with other applications, such as the vibrations of strings and sheets with specified mass distributions, the trajectories of certain celestial bodies, the oscillations of elliptic membranes and many other related topics. At a later stage, the Mathieu equation was also considered in connection with modes guided by elliptic waveguides, scattering by elliptic cylinders, the theory of frequency modulation, scattering by periodic crystal lattices and many other physical phenomena.

Most of the studies mentioned above were concerned with the purely periodic type of solutions of Mathieu and Hill's equations, while the nonperiodic solutions were analyzed less extensively. These nonperiodic functions relate to either stable or unstable physical configurations. Among the earlier investigators, Strutt[2] analyzed these functions, applied them to an atomic grating subject to a periodic sinusoidal potential and then briefly mentioned the significance and application of the Mathieu stability charts. These stability charts were considered in greater detail by van der Pol and Strutt[3] in connection with particles in force fields which are characterized by sinusoidal and rectangular periodic variations.

A direct application of these stability charts to waves in periodically-modulated media was made by Brillouin,[4] who also pointed out the relationship between these stability charts and the dispersion curves for the modulated media. The extensively-studied periodic solutions to Mathieu's equation are found to apply only at the band edges, while the nonperiodic functions describe the behavior within the pass and stop bands. A portion of the present paper also employs the stability charts to

[1] Lord Rayleigh, "On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure," *Phil. Mag.*, vol. 24, pp. 145–159; August, 1887.

[2] M. J. O. Strutt, "Zur Wellenmechanik des Atomgitters," *Ann. der Phys.*, vol. 86, no. 10, pp. 319–324; 1928.

[3] B. van der Pol and M. J. O. Strutt, "On the stability of the solution of Mathieu's equation," *Phil. Mag.*, vol. 5, pp. 18–38; January, 1928.

[4] L. Brillouin, "Wave Propagation in Periodic Structures," Dover Publications, Inc., New York, N. Y.; 1953.

determine the dispersion characteristics, but treats the problem in more detail and considers a number of additional aspects. This paper also includes detailed discussions on features which have not been previously treated, such as the properties of the various space harmonics and the total field behavior under various conditions. Explicit and simple analytic expressions for the dispersion characteristics and the field behavior in the range of small modulation amplitude are also given here.

The present study starts with the consideration of the wave equation in an infinite sinusoidally-modulated medium, and it is shown that, for E modes, this equation may be cast into the form of a Hill's differential equation; for H modes, on the other hand, the wave equation is a Mathieu differential equation, which is a particular case of Hill's equation. The discussion is then restricted to H modes since these are described by the simpler Mathieu equation. The properties of the Mathieu functions and the electromagnetic modes which they represent in the present case are discussed in Section III along with some examples of modes in bounded, rather than infinite, structures. The aspects of small modulation are treated at length in Section IV and analytical results are presented. The field within a unit cell of the stratified medium is calculated in Section V for both small and large modulations; in addition, the detailed field distribution within a waveguide filled with such a periodically-modulated medium is plotted for a variety of frequencies.

II. Formal Wave Solutions for H Modes

The geometry of the medium considered is shown in Fig. 1. This medium is assumed to extend to infinity in all directions and to possess a relative dielectric constant

$$\epsilon(z) = \epsilon_r\left(1 - M\cos 2\pi\frac{z}{d}\right) \tag{1}$$

where ϵ_r is the average value of $\epsilon(z)$ and M is termed the modulation index. The medium can therefore be visualized to consist of striations parallel to the xy plane and spaced a distance d apart.

The wave equations to be satisfied by the field solutions are then

$$\nabla^2\boldsymbol{E} + k_0{}^2\epsilon(z)\boldsymbol{E} = 0 \tag{2}$$

$$\nabla^2\boldsymbol{H} + k_0{}^2\epsilon(z)\boldsymbol{H} + \frac{\nabla\epsilon(z)}{\epsilon(z)} \times (\nabla \times \boldsymbol{H}) = 0 \tag{3}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are, respectively, the electric and magnetic field vectors, $k_0 = \omega(\mu_0\epsilon_0)^{1/2}$ is the intrinsic wave-number of free space, and a time dependence $e^{-i\omega t}$ is assumed.

The field may be represented in the form of superpositions of H and E modes, with

H modes: $\boldsymbol{E} = \boldsymbol{E}_y$; $H_y = E_x = E_z = 0$

E modes: $\boldsymbol{H} = \boldsymbol{H}_y$; $E_y = H_x = H_z = 0$.

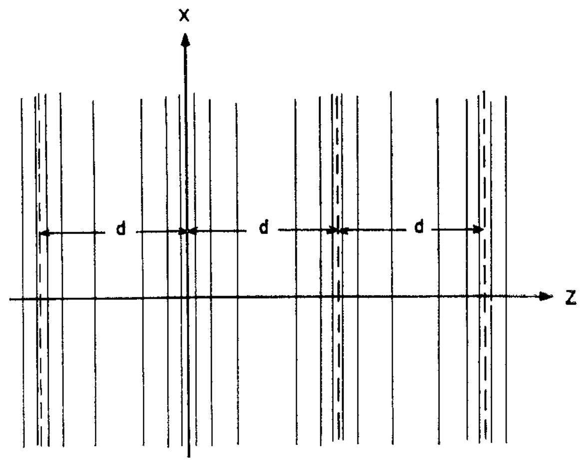


Fig. 1—Geometry of the stratified (modulated) medium.

Furthermore, it is assumed that no variation exists along the y coordinate ($\partial/\partial y \equiv 0$). Eqs. (2) and (3) then reduce to

$$\nabla^2 E_y + k_0{}^2\epsilon(z)E_y = 0 \tag{4}$$

$$\nabla^2 H_y + k_0{}^2\epsilon(z)H_y - \frac{1}{\epsilon(z)}\cdot\frac{\partial\epsilon(z)}{\partial z}\cdot\frac{\partial H_y}{\partial z} = 0. \tag{5}$$

Hence, (4) and (5) are, respectively, the wave equation for H and E modes. Since (5) has a term which is absent in (4), the latter equation is simpler and its solution is more straightforward. In the present paper the treatment is restricted to H modes.

Since (4) has coefficients which are functions of z only, one may assume solutions in the form

$$E_y(x, z) = Z(k_t; z)e^{ik_t x}. \tag{6}$$

From this separation of variables, one obtains from (4) and (1)

$$\left[\left(\frac{d}{dz}\right)^2 + k_0{}^2\epsilon_r - k_t{}^2 - k_0{}^2\epsilon_r M\cos 2\pi\frac{z}{d}\right]Z(k_t; z) = 0. \tag{7}$$

It is convenient, at this stage, to introduce the following notation:

$$\kappa_u{}^2 = k_0{}^2\epsilon_r - k_t{}^2 \tag{8}$$

$$q = \frac{M}{2}\cdot\left(\frac{k_0 d}{\pi}\right)^2\cdot\epsilon_r \tag{9}$$

$$x' = \pi\frac{x}{d}, \qquad z' = \pi\frac{z}{d}\,. \tag{10}$$

The quantity κ_u is recognized as the propagation wave-number in the z direction which would be present in a uniform medium of dielectric constant ϵ_r. When these changes in notation are substituted into (7), one finds

$$\left[\left(\frac{d}{dz'}\right)^2 + \left(\frac{\kappa_u d}{\pi}\right)^2 - 2q\cos 2z'\right]Z(k_t; z) = 0. \tag{11}$$

This last equation is the canonical form of Mathieu's differential equation; its solutions may be expressed in the Floquet form

$$Z(k_t; z) = e^{\pm i\kappa z} P(k_t; z) \tag{12}$$

where $P(k_t; z)$ is periodic in z with a periodicity of d. The term κ is termed the characteristic exponent and is a single-valued function of both κ_u and q; κ is clearly a propagation wavenumber for waves traveling in the $\pm z$ direction, and only the positive sign need be retained in (12) in order to determine the modes in the given infinite geometry. The periodic function $P(k_t; z)$ may next be expanded in the Fourier series

$$P(k_t; z) = \sum_{n=-\infty}^{\infty} a_n e^{in(2\pi/d)z} \tag{13}$$

where the coefficients $a_n = a_n(k_t)$ are dependent on the separation variable k_t.

One realizes that when no modulation is present $M=0$, and therefore q in (11) also vanishes. Then all the coefficients a_n vanish except a_0, and κ becomes $\kappa \equiv \kappa_u$; hence (8), *i.e.*,

$$\kappa_u^2 = k_0^2 \epsilon_r - k_t^2$$

represents the dispersion relation for the unmodulated medium. For finite values of M, on the other hand, the dispersion relation is more complicated and is obtained in the following manner.

Introducing (13) into the Mathieu equation (11), one obtains

$$\sum_{n=-\infty}^{\infty} \left[\left(\frac{\kappa_u d}{\pi}\right)^2 - \left(\frac{\kappa d}{\pi} + 2n\right)^2 - q(e^{2iz'} + e^{-2iz'}) \right] a_n e^{2inz'} = 0. \tag{14}$$

Since the above must hold for any value of z', (14) can be written as an infinite set of homogeneous equations in the form

$$a_{n-1} + D_n a_n + a_{n+1} = 0 \tag{15}$$

where

$$D_n = \frac{\left(\frac{\kappa d}{\pi} + 2n\right)^2 - \left(\frac{\kappa_u d}{\pi}\right)^2}{q} = \frac{\left(\kappa + \frac{2n\pi}{d}\right)^2 - \kappa_u^2}{M k_0^2 \epsilon_r / 2}. \tag{16}$$

Employing an iterative process on (15), one obtains the two continued fractions[5,6]

$$\frac{a_n}{a_{n-1}} = -\frac{1|}{|D_n} - \frac{1|}{|D_{n+1}} - \frac{1|}{|D_{n+2}} - \cdots \tag{17a}$$

$$\frac{a_n}{a_{n+1}} = -\frac{1|}{|D_n} - \frac{1|}{|D_{n-1}} - \frac{1|}{|D_{n-2}} - \cdots \tag{17b}$$

which, when combined, yield

$$D_n = \frac{1|}{|D_{n+1}} - \frac{1|}{|D_{n+2}} - \cdots + \frac{1|}{|D_{n-1}} - \frac{1|}{|D_{n-2}} - \cdots. \tag{18}$$

The last expression is the required dispersion relation since it is essentially an equation for $\kappa = f(\kappa_u, q)$. The continued fractions involved in (17) and (18) can be shown[6] to converge if $|D_n| \geq 2$ for $n > N$, where N is a finite integer. An inspection of (16) shows that this condition is always satisfied for the case considered here. The value of κ may therefore be computed for any given κ_u and q; a calculation of this type is given elsewhere.[7] When the appropriate value of κ is thus obtained, all the possible field solutions are known by means of (6), (12), (13) and (17).

Calculations for the dispersion relation (18) are very scarce in the literature. When $\kappa d/\pi = r$ ($r = 0, \pm 1, \pm 2, \cdots$), the solution given by (12) is termed a Mathieu function. These functions are of special interest for problems involving elliptic geometries and they have been extensively tabulated.[8] However, for $\kappa d/\pi$ not an integer or for complex values of this variable, very few tabulations have been carried out;[6,9] also, these calculations were performed for restricted ranges of the parameters q and κ_u, so that their application is somewhat limited.

III. Characteristics of the H Mode Solutions

In Section II the wave solutions for H modes were found to be of the form

$$E_y(x, z) = \sum_{n=-\infty}^{\infty} a_n e^{ik_t x} e^{i\kappa_n z} \tag{19}$$

where

$$\kappa_n = \kappa + \frac{2n\pi}{d} \tag{20}$$

and the coefficients a_n are determined from (17) in terms of a_0. The evaluation of a_0 itself is dependent on a normalization condition.

[5] N. W. McLachlan, "Theory and Application of Mathieu Functions," Oxford University Press, Oxford, England; 1951.
[6] J. Meixner and F. W. Schafke, "Mathieusche Funktionen and Spharoidfunktionen," Springer-Verlag, Berlin, Germany; 1954.
[7] T. Tamir, "Characteristic exponents of Mathieu functions," *Math. Comput.*, vol. 16, pp. 100–106; January, 1962.
[8] National Bureau of Standards, "Tables Relating to Mathieu Functions," Columbia University Press, New York, N. Y.; 1951.
[9] S. J. Zaroodny, "An Elementary Review of the Mathieu-Hill Equation of Real Variables Based on Numerical Solutions," Ballistic Research Lab., Aberdeen Proving Ground, Md. Memo. Rept. No. 878; April, 1955.

Eq. (19) is a solution of the wave equation (4) and therefore represents a mode of propagation in the z direction for the medium considered when the "transverse" wavenumber k_t is prescribed. Alternatively, when κ and d are prescribed, solution (19) represents a mode of propagation in the x direction (along the striations) with propagation wavenumber k_t. The former mode evidently consists of an infinite but discrete number of space harmonics, each such space harmonic being in the form of a plane wave propagating at an angle

$$\theta_n = \tan^{-1} \frac{\kappa_n}{k_t} \tag{21}$$

with respect to the xy plane, provided both κ_n and k_t are real. When either κ_n or k_t is imaginary, the waves have an amplitude which must decay in the z or the x direction, respectively. One also notes from (19) and (20) that the propagating or decaying character of the waves is the same for each and every space harmonic.

The relation between κ_u, q and κ can be illustrated in the form of a "stability diagram" which is customary in the study of the Mathieu equation.[4–9] This chart may be computed from relations given in the preceding section and described elsewhere.[7] The relation between the parameters κ_u, κ and q is then shown in Fig. 2.[10] The shaded areas in Fig. 2 are the so-called "stable regions" wherein κ is pure real, as indicated in the figure. The term "stable regions" refers to the fact that, if κ lies within such a region, the solutions of the Mathieu equations are bounded for any z. Outside the stable regions, κ is complex and its value is

$$\pm\kappa = \frac{m\pi}{d} + i\alpha \qquad (m = 0, 1, 2, \cdots) \tag{22}$$

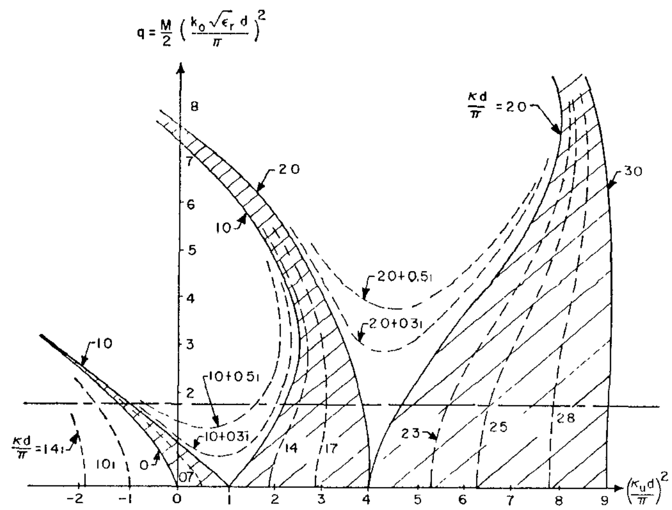


Fig. 2—Mathieu stability diagram. The curves in the unstable (unshaded) regions are approximate.

where m is the value of $|\kappa d/\pi|$ at the boundaries of the appropriate regions and α is pure real. The $\pm$ sign for κ in (22) is due to the fact that the curves in Fig. 2 essentially determine relationships for κ^2 rather than κ. These regions with complex values of κ are termed "unstable" in the literature since one of the values for κ yields solutions which are not bounded at infinity. It should be recognized, however, that the term "unstable" is not appropriate; in actual physical situations wherein the fields extend to infinite values of z, a radiation condition is required which then merely means that the unbounded solution is rejected and only the decaying one is retained.

It is noted that the value of κ for the *stable* regions is given by

$$m \leq \left| \frac{\kappa d}{\pi} \right| \leq m + 1 \qquad (m = 0, 1, 2, \cdots) \tag{23}$$

so that the value of m may label the appropriate regions, with $m=0$ denoting the region containing the origin, while increasing values of m refer to regions which are progressively removed from the origin.

To obtain some insight into the character of the modes given in (19), let us now look at a specific case as prescribed by specific values of ϵ_r, M, d and k_0, while k_t varies. It is then clear, from (9), that this specifies a value of q which is constant, and one therefore needs to consider points along a line parallel to the κ_u axis, such as the dot-dashed line shown in Fig. 2.

For very large real values of k_t, the relevant point for κ is in the unstable region to the left of the curve $\kappa=0$. Hence, from (22) with $m=0$, one has a field with an exponential term in z. In general, the solution in an unstable region may be written

$$\begin{aligned} Z(k_t; z) &= e^{-\alpha z} e^{im\pi(z/d)} P(k_t; z) \\ &= e^{-\alpha z} \hat{P}(k_t; z) \end{aligned} \tag{24}$$

where $\hat{P}(k_t; z)$ is a periodic function in z with period d or $2d$ for even or odd values of m, respectively. For simplicity, only the positive sign for κ is considered in (22), so that (24) refers to a wave progressing in the positive z direction; the expressions pertaining to waves progressing in the negative z direction may be derived by a suitable change in the sign for κ. It can be shown[4,5,11] that $\hat{P}(k_t; z)$ is either a pure real or a pure imaginary function, but is not complex. Consequently, the Fourier harmonics in $\hat{P}(k_t; z)$ couple in pairs and form a standing wave.

Hence, in all unstable regions, the modes are in the form of an exponentially damped standing wave in the z direction and are propagating in the x direction only. This type of wave therefore consists of space harmonics which couple in pairs so that they propagate along the

[10] The stability diagram is seen to consist of curves drawn in a q vs $(\kappa_u d/\pi)^2$ with $\kappa d/\pi$ itself as a variable parameter. To abbreviate the notation, this diagram will be referred to, in the discussion below, as the q vs κ_u [rather than $(\kappa_u d/\pi^2)$] diagram.

[11] T. Tamir, H. C. Wang and A. A. Oliner, "Wave Propagation in Sinusoidally Stratified Dielectric Media," Dept. of Electrophysics, Polytechnic Inst. of Brooklyn, N. Y., Research Rept. No. PIBMRI-1184-63; 1963.

striations. It is noted also that the wavenumber for each harmonic is given from (20), with only the positive sign being retained, by

$$\kappa_n = (2n + m)\frac{\pi}{d} + i\alpha, \tag{25}$$

so that the attenuation coefficient α is the same for every space harmonic.

When the value of k_t is decreased, one reaches a value for which the dot-dashed line intersects the $\kappa = 0$ line; the field then has the form of a pure standing wave with no decay along z, and with propagation still along the striations.

As k_t further decreases, the first "stable" region is reached in which κ becomes real; the harmonics then propagate in both the x and z directions at angles determined by (21). By still further decreasing k_t, one again crosses unstable and stable regions and the waves have the properties of the cases previously discussed. The number of regions thus crossed is dependent on the value of $k_0\sqrt{\epsilon_r}$ since, as $k_t = 0$, $\kappa_u^2 = k_{0t}^2$, and the point of interest cannot go beyond this value unless k_t becomes imaginary.

In order to obtain a clearer picture of the above features, it is worthwhile to consider the waves which would be excited in the periodically-modulated medium by some typical sources. The following two cases are physically significant.

A. Line Source Excitation

An electric current line source is assumed to be present parallel to the y axis and located at $x = 0$ and $z = h$ in the infinite medium of Fig. 1. The field of such a source may then be expressed by the Fourier integral representation

$$E(x, z) = \int_{-\infty}^{\infty} f(k_t)e^{ik_t x} \sum_{n=-\infty}^{\infty} a_n e^{i\kappa_n|z-h|} dk_t \tag{26}$$

where, to satisfy the radiation condition, the $+$ sign must be taken in (22); $f(k_t)$ is an amplitude function in terms of k_t. Both uniqueness and completeness of the above representation are satisfied if the integration in (26) is carried out along the real k_t axis, in agreement with the theory of Fourier transform representations. The field is therefore properly visualized as being made up of the waves previously considered, when k_t varies between infinity and zero.

One therefore concludes that the field of a line source consists of a continuous spectrum of modes characterized by the transverse wavenumber k_t. Each mode consists of space harmonics in the form of plane waves. All of the space harmonics associated with any one particular mode either propagate along the striations or travel at various angles with respect to these striations. In the latter case, the plane waves are homogeneous; in the former case, the fields decay away from the striations closest to the source, this decay being the same for all the harmonic components.

The total energy radiated by a line source therefore contains a part which is guided along the striations; consequently, the region formed by the striations closest to the source acts as a duct. This is not unexpected, since it is known that a dielectric slab may guide surface waves. However, in the case considered here, the waves which are guided along the striations form part of a continuous set rather than the discrete surface wave contributions that occur on a single dielectric slab.

B. Plane Wave Excitation

One cannot assume the existence of a single homogeneous plane wave in the modulated medium since such a wave is not a solution of the pertinent wave equation (4). Nevertheless, one may consider that such a plane wave is incident on a semi-infinite medium, as shown in Fig. 3, and thereby excites a field in the medium considered. The modulated medium is taken to extend only for positive values of z, and a plane wave is assumed to be incident at an angle θ on the xy plane interface from a medium with a relative dielectric constant ϵ_1. Since H modes are considered, the electric vector E is taken parallel to the interface between the two media at $z = 0$. No attempt is made here to solve the complete problem; only the character of the field in the modulated medium is considered.

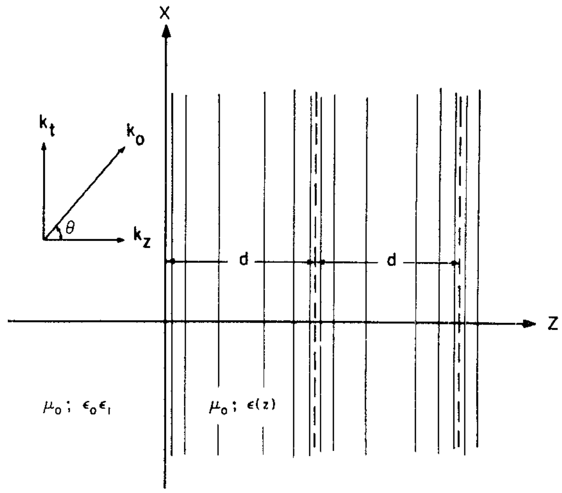


Fig. 3—Geometry of the semi-infinite modulated medium.

In contrast to the line source problem, the present situation has a fixed value of k_t given by

$$k_t = k_0\sqrt{\epsilon_1}\sin\theta \tag{27}$$

which yields, from (8),

$$\kappa_u = k_0(\epsilon_r - \epsilon_1 \sin^2\theta)^{1/2}. \tag{28}$$

Let us introduce the angle ϕ shown in Fig. 4 as

$$\tan\phi = q\left(\frac{\pi}{\kappa_u d}\right)^2 = \frac{M}{2}\cdot\frac{1}{1 - \frac{\epsilon_1}{\epsilon_r}\sin^2\theta}, \tag{29}$$

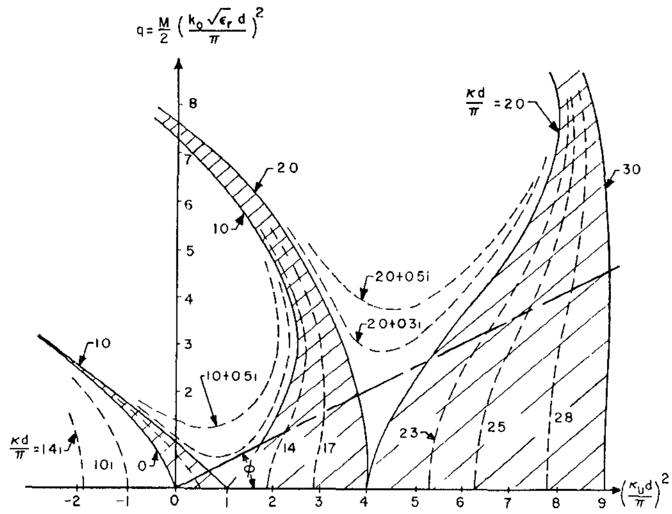


Fig. 4—Graphical construction to obtain dispersion curves in terms of a given incidence angle.

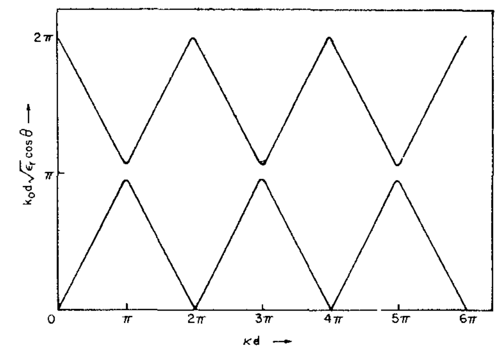


Fig. 5—Dispersion curves (Brillouin diagram) for the semi-infinite medium: $\epsilon_1=\epsilon_r$; $\tan\phi=M/2\cos^2\theta=0.1$.

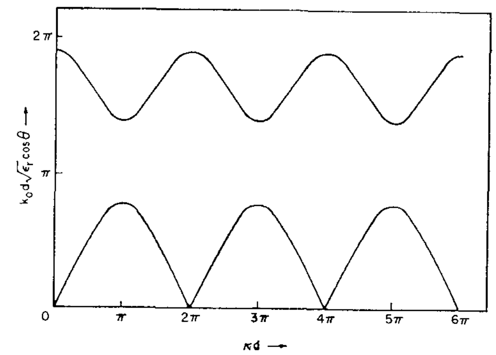


Fig. 6—Dispersion curves (Brillouin diagram) for the semi-infinite medium: $\epsilon_1=\epsilon_r$; $\phi=\tan^{-1}(M/2\cos^2\theta)=30°$.

using (28) and definition (9) for q. The locus of wavenumbers κ which are excited in the modulated medium is therefore given by a straight line through the origin and at an angle ϕ with respect to the $(\kappa_u d/\pi)^2$ axis. Then, as frequency varies while M and the incidence angle θ remain constant, κ_u and q vary along this straight line and the appropriate value for κ may be read off the q vs κ_u diagram of Fig. 4.

One may then obtain a dispersion curve for any specified values of M and θ. Real values of κ will correspond to solutions which propagate across the striations, while complex values of κ correspond to exponential decay, thus resulting in pass and stop bands, respectively, for propagation perpendicular to the striations.

The following cases are then relevant:

1) $M=0$: When no modulation is present, $\phi=0$, the locus for the dispersion curve runs along the positive real κ_u^2 axis and no stop bands occur for $\epsilon_r\geq\epsilon_1$. This result is obvious since, if $M=0$, the medium to the right of the interface is an ordinary dielectric into which waves are propagated by ordinary refraction of the incident wave. For $\epsilon_r<\epsilon_1$, certain angles θ will make the expression for $\tan\phi$ in (29) negative. Then no propagation occurs for any frequency since the dispersion curves run along the negative real κ_u^2 axis. This situation obviously occurs at and above the critical optical angle for which total reflection of the incident wave is obtained.

In order to disregard the possibility of total reflection, the following cases are taken for $\epsilon_1\leq\epsilon_r$ only:

2) $M\ll 1$: The angle ϕ is then very small and one obtains broad pass bands and narrow stop bands, as shown in Fig. 5. The latter bands occur in the neighborhood of frequencies whose free-space wavelength λ_0 is given, from (28), by

$$\lambda_0\cong\frac{2d}{m}\sqrt{\epsilon_r-\epsilon_1\sin^2\theta}\qquad(m=1,2,3,\cdots).\tag{30}$$

This result is analogous to the Bragg effect in crystal lattices.

3) $M<1$: The angle ϕ is no longer small and the stop bands increase in width, as shown in Fig. 6 for a value of $\phi=30°$. It is realized that for ordinary dielectrics $\epsilon(z)\geq 1$, which yields, from (1),

$$M\leq\frac{\epsilon_r-1}{\epsilon_r}\cdot\tag{31}$$

This condition implies a maximum value of ϕ for any incidence angle θ and dielectric constant ϵ_1.

It is recognized that for plasma media $\epsilon(z)$ may be less than unity or even negative; however, these values of the relative dielectric constant are a function of frequency. The above geometrical construction for finding the dispersion curves has assumed that $\epsilon(z)$ is independent of frequency so that, for plasma media or other dispersive media, this procedure would have to be modified.

4) $M\neq 0$; $\theta\rightarrow\pi/2$: As the angle of incidence θ reaches a value of $\pi/2$, the angle ϕ takes on the maximum of $\phi=\pi/2$. The pass bands then become very narrow while the stop bands are relatively very broad.

The above results show that the dispersion curves for plane wave excitation are similar to those for customary periodic transmission lines. It should be noted that the dispersion curves for the medium discussed here may be found exactly from the q vs κ_u diagram; for most examples of periodic transmission lines, however, the dispersion curves are found only by assuming certain simplifying approximations which are necessary in view of the greater complexity of the problem.

C. "*Waveguide*" *Excitation*

In Sections III-A and III-B, the waves considered were those for which the transverse wavenumber k_t is a function of frequency. In waveguide practice, the usual approach is to have k_t prescribed by the boundary conditions in the cross section. For an infinite medium, the wavenumber k_t is continuously variable and the approach in Sections III-A and III-B is appropriate. Nevertheless, it is of interest to consider the case in which the modulated medium is contained between two conductors, both parallel to the yz plane, as shown in Fig. 7. The propagation direction is clearly the z direction, and the medium is uniform in the xy cross section, with the same value of k_t applicable for all values of z. Since the electric field is parallel to the y direction, the field distribution and the propagation characteristics are unaffected by the placement of two additional perfectly conducting planes at right angles to the first two, forming a rectangular waveguide. The pertinent mode set in this rectangular waveguide is that of the H_{n0} modes. It is shown below that some propagation features appear here that are absent for the infinite medium.

One recognizes, for the geometry of Fig. 7, that the wavenumber k_t cannot vary continuously but has only the discrete values

$$k_{tr} = \frac{r\pi}{L} \qquad (r = 0, 1, 2, \cdots). \tag{32}$$

To obtain dispersion curves for this case it is easily seen that, as frequency varies, the quantity

$$\tan \phi' = \left(\frac{\pi}{d}\right)^2 \frac{q}{\kappa_u^2 + k_{tr}^2} = \frac{M}{2} \tag{33}$$

is a constant for any given M. Eq. (33) is then the equation of straight lines as shown in Fig. 8. This case is similar geometrically to that for plane wave excitation, discussed previously, except that the straight line is shifted from the origin. In addition, an entire family of such lines must be considered in order to accommodate the various values of k_{tr}, all these lines being parallel to each other.

The values for the dispersion curves are now read off from Fig. 8 in the same manner as those obtained for plane wave excitation. It is clear that now the first pass band starts at a finite, nonzero frequency. When $M \neq 0$, this is the cutoff frequency of the unmodulated waveguide. For $M \neq 0$, the first pass band starts at a frequency which is *lower* than the cutoff value for the unmodulated waveguide. This feature is present for all the modes and is determined only by the shape and slope of the $\kappa = 0$ curve in the q vs κ_u diagram, as may be seen by inspection of Fig. 8.

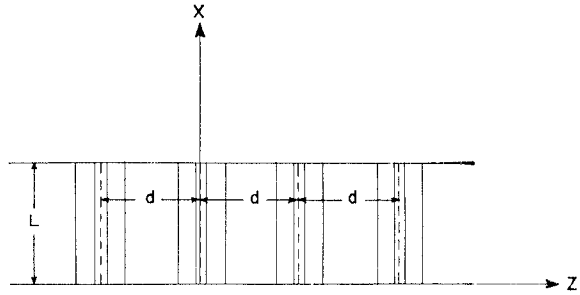


Fig. 7—Rectangular guide filled with a longitudinally-modulated dielectric.

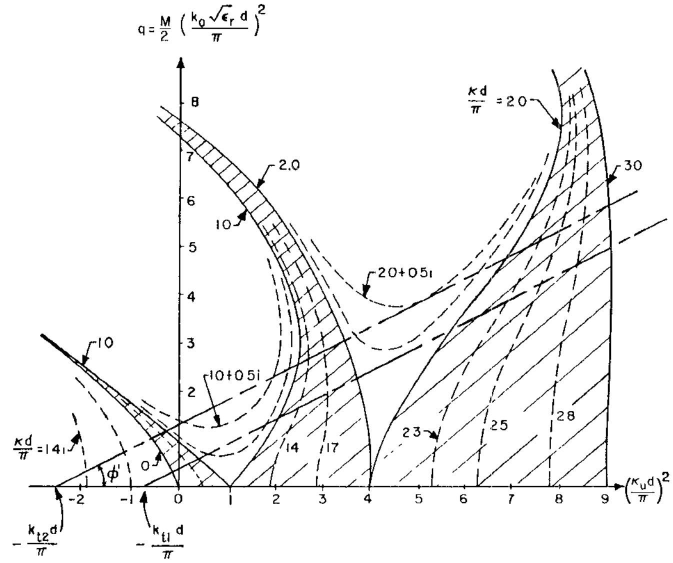


Fig. 8—Graphical construction to obtain dispersion curves for the waveguide filled with a modulated medium.

The above behavior is illustrated in Figs. 9 and 10, which show dispersion curves for the first mode ($k_t = k_{t1}$) for various values of M. It is clear that the arguments associated with (31), limiting the value of M, are also applicable here, and that ϕ' is correspondingly limited. It is then obtained that, in the limit of ϵ_r very large, $M = 1$ and the maximum value of the angle ϕ' is given by

$$\tan \phi'_{\max} = \tfrac{1}{2}. \tag{34}$$

The dispersion curves for this specific angle are given in Fig. 10(a).

As noted above, when $M \neq 0$ propagation is possible at frequencies for which the guide containing an *unmodulated* medium would be below cutoff. This behavior is evidenced in Fig. 10(b) in which the variation of κ_u vs κ is shown. The lower part of the curve then corresponds to propagation in the *modulated* guide for which κ_u is imaginary, *i.e.*, if no modulation were present the wave would be below cutoff.

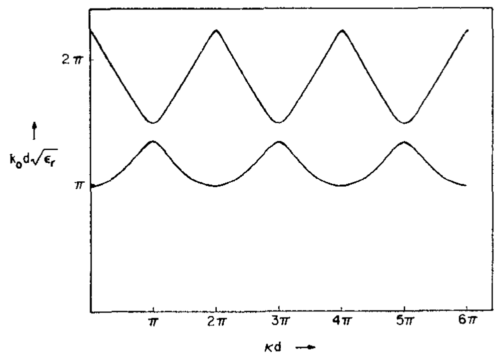


Fig. 9—Dispersion curves (Brillouin diagram) for the waveguide filled with a modulated medium: $M=0.2$; $k_t d=\pi$.

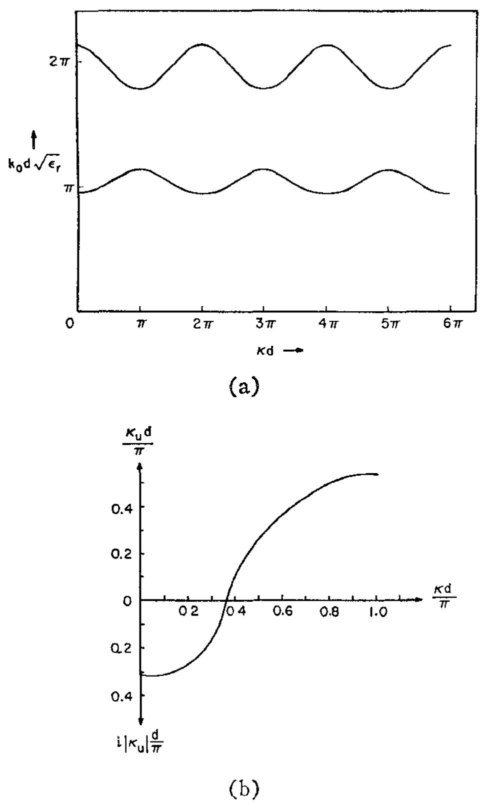


Fig. 10—Dispersion curves for the waveguide filled with a modulated medium: $M=1.0$; $k_t d=\pi$. (a) Brillouin diagram. (b) Propagation wavenumber κ in the waveguide containing the modulated medium vs propagation wavenumber κ_u in the unmodulated waveguide.

IV. Analytic Results for Small Modulation

So far, the entire discussion of the wave features for a modulated dielectric medium followed from *graphical* constructions based on the q vs κ_u diagram which indirectly characterizes the functional properties of the modes in such a medium. One may also wish to obtain these results in an *analytic* form; however, because of the Mathieu functions involved, a simple analytic result is not obtainable unless some simplifying restrictions are imposed.

In this section certain restrictions are stipulated so that practical analytic expressions are obtained for the dispersion relation and the amplitudes of the various space harmonics. One may then derive complete analytic solutions for the problems of an incident plane wave on the semi-infinite modulated dielectric, treated in Section III-B, and for the waveguide containing that dielectric discussed in Section III-C. The restrictions are, however, of such a nature that these results apply to the pass bands only, and then not too close to the band edges; in addition, the modulation index must be restricted to sufficiently small values. For more complete results, appropriate for less restrictive conditions, see Wang and Tamir.[12]

A. Approximate Relations for the Pass Bands

An inspection of the Mathieu stability diagram (Fig. 2) reveals that, within the stable regions (pass bands) and for small values of q, κ is real and very nearly equal to κ_u. One may then write

$$\frac{\kappa d}{\pi} = \frac{\kappa_u d}{\pi} + \Delta \tag{35}$$

where Δ is a very small quantity. Then, for D_n of (16),

$$D_n \cong \frac{4n(\kappa_u d/\pi + n)}{q} \qquad (n \neq 0) \tag{36}$$

$$D_0 \cong \frac{2\kappa_u d}{q\pi}\Delta. \tag{37}$$

Since q was assumed to be very small, one has that

$$|D_n| \gg 1 \qquad (n \neq 0), \tag{38}$$

so that the dispersion relation (18) becomes, to a first approximation,

$$D_0 \cong \frac{1}{D_1} + \frac{1}{D_{-1}}\cdot \tag{39}$$

Result (39) implies that one need retain only three space harmonics ($n=0, \pm 1$) in the dispersion relation. By introducing (35), (36) and (37) into (39) and retaining first terms only, one obtains this dispersion relation in the form

$$\frac{\kappa}{\kappa_u} = 1 + \frac{1}{1-(\kappa_u d/\pi)^2}\left(\frac{\pi}{\kappa_u d}\cdot\frac{q}{2}\right)^2. \tag{40}$$

This result for the dispersion relation is valid in the pass bands only, and not too close to the band edges; its accuracy is discussed quantitatively by Wang and Tamir.[12]

One recalls that q, which is defined by

$$q = 2M\left(\frac{d}{\lambda}\right)^2 \epsilon_r, \tag{41}$$

is assumed to be small for the above calculations. One sees that this condition is satisfied either by small values

[12] H. C. Wang and T. Tamir, "Closed form dispersion relations for a sinusoidally stratified medium," to be published.

of M or by small values of d/λ. Hence, if $d/\lambda \ll 1$, the modulation in the dielectric may be quite appreciable.

Since, to the approximation above, only the fundamental and the $+1$ and -1 space harmonics need be retained, the total field of a mode characterized by k_t is given by the first three terms in (19), namely,

$$E_y(x, z) = a_0 e^{i(k_t x + \kappa z)} \left[1 + \frac{a_1}{a_0} e^{i(2\pi/d)z} + \frac{a_{-1}}{a_0} e^{-i(2\pi/d)z} \right]. \tag{42}$$

In view of condition (38), only the first term need be retained in (17), so that one obtains for the ratio of amplitudes

$$\frac{a_1}{a_0} = -\frac{q}{4} \frac{1}{(1 + \kappa_u d/\pi)}, \tag{43}$$

$$\frac{a_{-1}}{a_0} = -\frac{q}{4} \frac{1}{(1 - \kappa_u d/\pi)}. \tag{44}$$

When (43) and (44) are substituted into (42) and terms are combined, the expression for the field becomes

$$E_y(x, z) = a_0 \left[1 - \frac{q}{2} \cdot \frac{\cos \frac{2\pi}{d} z - i \frac{\kappa_u d}{\pi} \sin \frac{2\pi}{d} z}{1 - (\kappa_u d/\pi)^2} \right] \cdot e^{i(k_t x + \kappa z)} \tag{45}$$

where κ is given by (35) and (40).

With no modulation present, $\kappa = \kappa_u$ and $q = 0$ so that (45) is reduced to the simple form of a plane wave traveling at a certain angle. When the medium is perturbed by the introduction of a small modulation, the change is reflected in the term containing q in (45). The perturbation may then be regarded as a small sinusoidal variation of the amplitude of the unperturbed wave. This variation is the same within every cell of period d and is the form of a sinusoidal curve which is shifted from the origin. The actual shape of the field within a cell is discussed in Section V, in which the field is also calculated numerically.

One notes that, for given parameters ϵ_r, M, k_t and ω, the solution for the field is generally obtained from (45) since the other parameters (κ_u and q) are given in terms of the prescribed constants. The value of a_0 is determined from a normalization condition or may be arbitrarily set to equal unity. Of course, the result thus obtained is valid only if the prescribed parameters ϵ_r, M, k_t and ω satisfy the restriction regarding small values of q in (41).

It is emphasized that all of the above results hold within the pass bands only and are not applicable to the stop bands. For the latter, κ is complex and it may be shown[5,6,11,12] that the amplitude of some particular harmonic is always equal to that of the fundamental wave (a_0) in the stop band; large errors may then result if only the 0 and ± 1 space harmonics are retained. For approximations and analytic results for the band edges and for the stop bands, see Wang and Tamir.[12]

B. Plane Wave Excitation

Result (45) may now be applied in order to obtain a complete solution to the problem of Section III-B in which a plane wave was incident from an unmodulated medium upon a modulated one, as shown in Fig. 3.

In this case, the boundary condition at $z = 0$ is satisfied by a single mode specified by (27), so that the field in both regions is completely determined by a reflection coefficient Γ at the interface. This reflection coefficient is given by

$$\Gamma = \frac{Z_2(0) - Z_1}{Z_2(0) + Z_1} \tag{46}$$

where Z_1 is the characteristic impedance of a transmission line in the z direction representing a wave traveling in the unmodulated medium ($z < 0$), and $Z_2(0)$ is the impedance loading that line at $z = 0$. Recalling that we are dealing with H modes, one has

$$Z_1 = \frac{\omega\mu}{k_z} = \frac{\omega\mu}{k_0 \sqrt{\epsilon_1} \cos\theta}. \tag{47}$$

The impedance $Z_2(0)$ may also be viewed as a characteristic impedance for the modulated medium, but associated with a particular choice of unit cell, *i.e.*, the interface between the two media specifies one end of the unit cell (the other is determined by the periodicity). A different location of the interface (with respect to the modulated medium) would result in a different equivalent characteristic impedance. We recognize that in the modulated region there exists only a forward-traveling wave, so that

$$Z_2(z) = -\frac{E_{y2}(x, z)}{H_{x2}(x, z)} \tag{48}$$

where the subscript 2 indicates the modulated region ($z > 0$) and the negative sign arises because of the coordinate system chosen. The magnetic field $H_{x2}(x, z)$ is given by

$$H_{x2}(x, z) = -\frac{1}{i\omega\mu} \frac{\partial E_{y2}(x, z)}{\partial z}. \tag{49}$$

The electric field $E_{y2}(x, z)$ in the modulated region may be written in the form (45) in which only terms to the order of q were retained. Hence, we find from (45), (48) and (49) that

$$Z_2(z) = \frac{\omega\mu}{\kappa_u} \cdot \left[1 + \frac{\cos \frac{2\pi}{d} z - \frac{i\kappa_u d}{\pi} \sin \frac{2\pi}{d} z}{(\kappa_u d/\pi)^2 - 1} \cdot q \right] \tag{50}$$

where terms to the order of q only were retained. One also notes from (40) that κ and κ_u differ by a term in q^2, so that they were taken as $\kappa = \kappa_u$ when obtaining the result of (50).

It is observed that $Z_2(z)$ is approximately equal to the characteristic impedance $\omega\mu/\kappa_u$ of an unmodulated medium since the second term containing q is small. To find the reflection coefficient in (46) one needs

$$Z_2(0) = \frac{\omega\mu}{\kappa_u}\cdot\left[1 + \frac{q}{(\kappa_u d/\pi)^2 - 1}\right]. \tag{51}$$

When the value of $Z_2(0)$ is introduced into (46), the field may be found everywhere. Hence, to within the approximations considered here, one can obtain a complete solution for the problem of a plane wave incident on a semi-infinite modulated dielectric.

In particular, it is interesting to find the reflection coefficient when the dielectric constant ϵ_1 for the unmodulated medium is equal to the average dielectric constant ϵ_r in the modulated one. Then, for $\epsilon_1 = \epsilon_r$, one has

$$\kappa_u = k_z = k_0\sqrt{\epsilon_1}\cos\theta, \tag{52}$$

which yields a reflection coefficient

$$\Gamma = \frac{q/2}{(\kappa_u d/\pi)^2 - 1}. \tag{53}$$

Hence, the reflection from the entire semi-infinite modulated region is of the order of q.

We recognize, from definition (1) for the variation of the dielectric constant in the modulated medium, that the reflection coefficient solved for above corresponds to an interface described by Fig. 11(a). If, on the other hand, we were interested in an interface of the form shown in Fig. 11(b), it would be necessary to evaluate $Z_2(z)$ in (50) at $z = d/4$ rather than at $z = 0$. For such an interface, then, we find

$$Z_2(d/4) = \frac{\omega\mu}{\kappa_u}\left[1 + \frac{i\dfrac{\kappa_u d}{\pi}\cdot q}{1 - (\kappa_u d/\pi)^2}\right] \tag{54}$$

and

$$\Gamma = \frac{\dfrac{i}{2}\cdot\dfrac{\kappa_u d}{\pi}\cdot q}{1 - (\kappa_u d/\pi)^2} \tag{55}$$

when $\epsilon_1 = \epsilon_r$. At low frequencies, where $\kappa_u d$ is appreciably smaller than π, the reflection from the interface of Fig. 11(b) is seen to be less than that for Fig. 11(a), in agreement with intuition.

If the dielectric constant variation were as in Fig. 11(a), but inverted at the interface, we would find that the reflection coefficient magnitude is the same as that for (53), but that the phase is different by π, as we might expect. It is evident that the interface may be chosen to correspond to any point in the modulated medium, and that the above analysis can be extended in straightforward fashion to include the case of an arbitrary slab of this modulated medium.

The above results also hold for the case of the waveguide containing a modulated dielectric which was considered in Section III-C, by simply taking k_t as the actual cutoff wavenumber in the waveguide rather than the value used in (27). One can therefore solve the problem of wave propagation in such a waveguide for infinite, semi-infinite or finite lengths of the modulated medium.

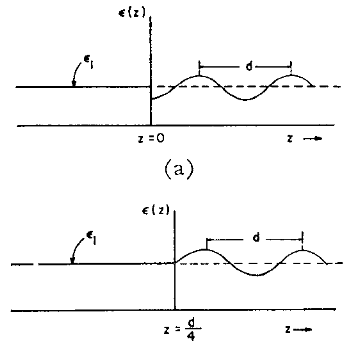


Fig. 11—Variation of the dielectric constant near the interface of the semi-infinite modulated medium. Two different interface situations are shown.

V. The Field Distribution within a Unit Cell

Most of the above discussions were concerned with the over-all, or macroscopic, propagation characteristics of waves in these modulated media. In order to obtain a more complete picture of the properties of these waves, this section considers the detailed field distribution within a unit cell of the sinusoidally-modulated medium. These calculations show that variation in the form of the fields as the frequency is changed and demonstrate that the field distribution need not follow the spatial variation in the dielectric constant.

In order to determine the variation of the electric field within a cell of length d in the sinusoidally-modulated medium, a specific dielectric is chosen, characterized by arbitrary but constant values of ϵ_r, d and M, with the frequency ω taken as the variable. To gain some insight into the shape of the field for a given mode, we shall first discuss certain relations between the amplitudes of the space harmonics and then explore the field shape in the range of small modulation index M. We will find that the field distribution varies considerably from one pass band or stop band to another. Because of the availability of relationships between the harmonic amplitudes in the stop bands, we will first examine the field behavior there and then proceed to the pass bands.

For an unmodulated medium ($M = 0$), only the fundamental space harmonic is present while all the other harmonics are zero ($a_n = 0$ for $n \neq 0$). For small modulations, all the harmonics are, in general, nonvanishing. In the stop bands (unstable regions), it can be shown[11,12] that the fundamental and the $-m$th harmonic are equal in amplitude in the mth stop band. In addition, numerical analysis shows that the other space harmonics decrease in strength as their harmonic number n is further removed from either the fundamental or $-m$th harmonic, while their amplitudes are equal in pairs; *i.e.*,

$$|a_n| = |a_{-(m+n)}|. \tag{56}$$

These properties are summed up in Fig. 12 which distinguishes between the following two cases: 1) For m even, the harmonics are equal in magnitude in pairs, except for the $-m/2$ harmonic which does not couple to any other one. This harmonic is zero in magnitude on the left edge of the unstable region in the q vs κ_u diagram and is finite on the right edge (these correspond, respectively, to the lower and upper edges of the stop band on a Brillouin diagram). 2) For m odd, the same behavior is present except that there is no uncoupled harmonic and none of the harmonics has a vanishing amplitude.

The electric field may be written as

$$E(z) = e^{i\kappa z} \sum_{n=-\infty}^{\infty} a_n e^{2in\pi(z/d)} = A(z)e^{i\phi(z)} \tag{57}$$

where $A(z)$ and $\phi(z)$ are, respectively, the amplitude and phase shift of $E(z)$. One then obtains for the amplitude

$$\begin{aligned} |A(z)|^2 &= \left[\sum_{n=-\infty}^{\infty} a_n e^{2in\pi(z/d)}\right] \\ &\cdot \left[\sum_{n=-\infty}^{\infty} a_n^* e^{-2in\pi(z/d)}\right] \cdot e^{-2\alpha z} \\ &= |A_p(z)|^2 e^{-2\alpha z} \end{aligned} \tag{58}$$

where $A_p(z)$ is the periodic part of $A(z)$.

In the first unstable region ($m=0$), the two sets of lines between $n=0$ and $n=-m$ in Fig. 12(a) coalesce, with the result that the $n=-m=0$ harmonic results in a relatively large constant value for $|A_p(z)|$ and the $+1$ and -1 harmonics yield a small sinusoidal wave of period d superimposed on this constant value. For small M, the higher harmonics add insignificant contributions.

For all the other unstable regions ($m>0$), the dominant terms in (58) are obtained as follows:

$$\begin{aligned} |A_p(z)|^2 &= [a_0 + a_{-m}e^{-2im\pi(z/d)}][a_0^* + a_{-m}^* e^{2im\pi(z/d)}] \\ &= 2|a_0|^2 \\ &\cdot \left\{1 + \cos\left[\frac{2m\pi}{d}z + \arg\left(\frac{a_{-m}}{a_0}\right)\right]\right\}. \end{aligned} \tag{59}$$

One has again a constant term and, in addition, a sinusoidal wave of period d/m. Also, since $|a_{-m}| = |a_0|$, as shown in Fig. 12, the amplitude vanishes m times within a period d. The phase term arg (a_{-m}/a_0) varies as one goes from one edge of the stop band to the other, so that the field pattern within the cell would seem to shift across the cell as a function of frequency within the stop band. The next better approximation would include the $n=\pm 1$ and $n=-m\pm 1$ harmonics which yield three sinusoidal variations [of period d, $d/(m+1)$ and $d/(m-1)$]. To the periodic variation of $A_p(z)$, one has to add the exponential decay of $e^{-\alpha z}$ in order to obtain the actual variation of $A(z)$.

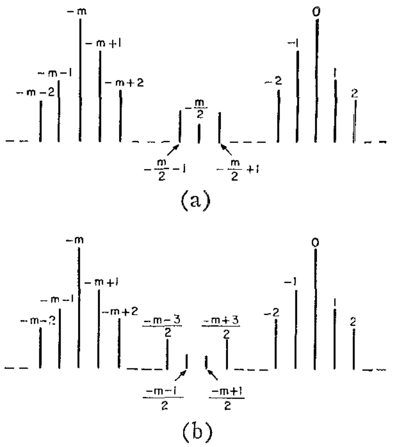


Fig. 12—Amplitudes of the space harmonics in an unstable region (stop band). (a) m even. (b) m odd.

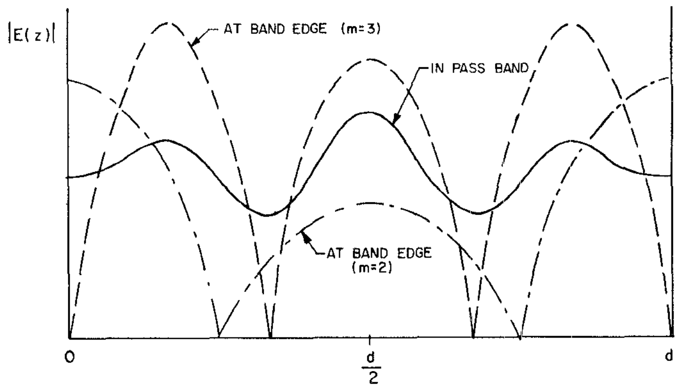


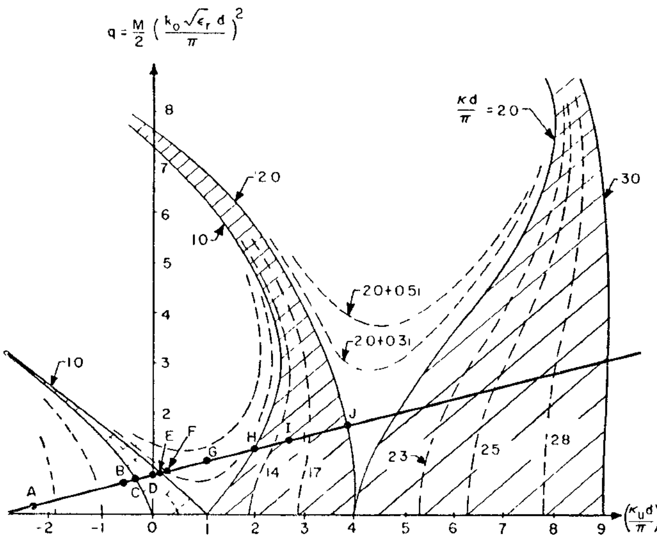
Fig. 13—Field distribution for the third pass band or stable region (located between the unstable regions characterized by $m=2$ and $m=3$).

In the stable regions (pass bands), noting that all a_n are pure real and that $\alpha=0$, one obtains for (58)

$$|A(z)|^2 = \sum_{n=-\infty}^{\infty} a_n \sum_{r=-\infty}^{\infty} a_{n-r} \cos 2\pi(n-r)\frac{z}{d}. \tag{60}$$

This is an even periodic function z; however, the various harmonics do not pair off in a manner similar to that of the unstable regions described in Fig. 12. Since $|A(z)|$ still possesses the properties discussed above at the edges of the pass band, we recognize that as one goes through the mth pass band from one edge to the other, the shape of the field must vary continuously from a periodicity of $d/(m-1)$ to a periodicity of d/m. This behavior is illustrated in Fig. 13.

Although the discussion above was carried out on the assumption that M is small, the periodicity features must be present even for large values of M. However, the shape of the field would then differ markedly from the simple constant and sinusoidal variation obtained in (59) since the additional harmonics may affect this basic shape considerably.



	q	$(\kappa_u d/\pi)^2$	$\kappa d/\pi$
A	0.2	-2.4	1.55i
B	0.7	-0.4	0.46i
C	0.735	-0.26	0.0
D	0.8	0.0	0.677
E	0.825	0.1	1.0
F	0.83	0.12	1-0.105i
G	1.05	1.0	1-0.47i
H	1.32	2.08	1.0
I	1.48	2.72	1.5
J	1.738	3.752	2.0

Fig. 14—Parameters for the computed fields. Points A to J relate to the fields shown in Figs. 15 and 16.

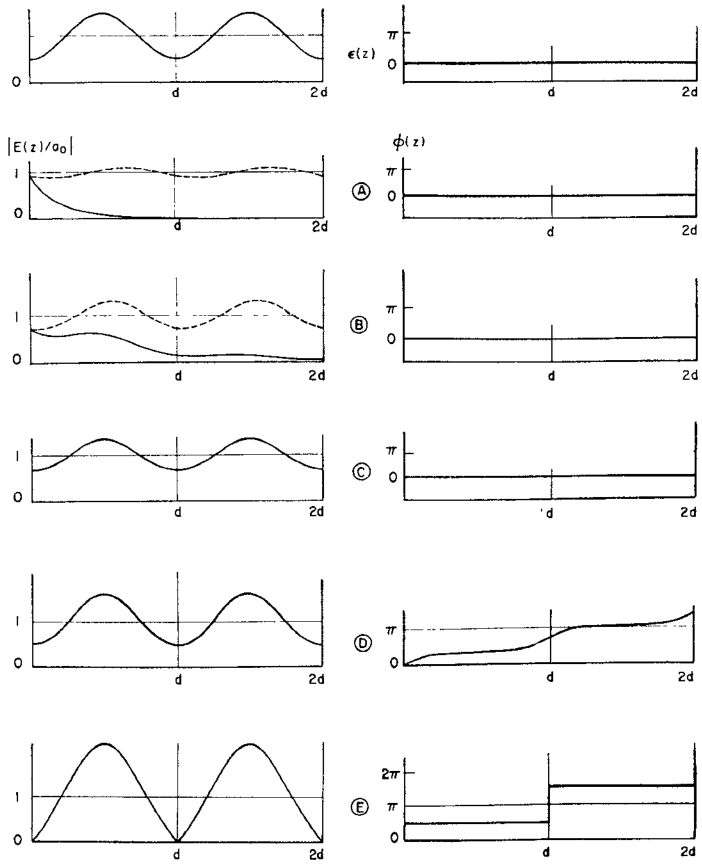


Fig. 15—Variation of the field in the first unstable (stop band) and the first stable (pass band) regions. Situations A to E are defined in Fig. 14. The $\epsilon(z)$ curves represent the actual variation in dielectric constant. Full and dashed lines refer, respectively, to $|A(z)|$ and $|A_p(z)|$ [see (58)].

To illustrate the various features discussed above, the shape of the electric field was calculated for a loaded waveguide of the type described in Section III-C. The parameters chosen and the various frequencies for which the calculations were made are shown in Fig. 14. The results obtained are illustrated in Figs. 15 and 16; these figures show the variation of the electric field as the frequency is varied so that the particular mode starts in the first unstable region and goes through stable and unstable regions up to the farther edge of the second stable region. The regions correspond physically to the frequency range below cutoff of the loaded waveguide, the first pass band, the first stop band, and the second pass band. In the unstable regions (below cutoff and in the stop band), the amplitude distribution shown corresponds to the periodic part $|A_p(z)|$ of the electric field amplitude; the total amplitude $|A(z)|$ is represented by the dashed lines.

One notes from these figures that the field changes continuously as the point of operation goes through the various regions; the general character of the field is clearly in agreement with the above qualitative analysis. It is interesting to note that although the modulation index chosen was not small ($M=0.5$), the features discussed previously are nevertheless strongly present.

As predicted, we find that the amplitude within the stop band (diagrams E, F, G and H of Fig. 16) goes through zero once (since $m=1$) within a period d, and that within this stop band the pattern essentially maintains its form but shifts across the cell as we go from one edge of the stop band to the other. At H, additional space harmonics become significant, thus altering the field distribution somewhat. The same behavior is also present within the below cutoff region (diagrams A and B of Fig. 15), but it is less noticeable there. The approximate analysis involving only the dominant space harmonics yields for the below cutoff region a constant amplitude only (since $m=0$); the sinusoidal variation is due to the next higher ($n=\pm 1$) space harmonics. Within the pass bands the behavior also corresponds to that predicted by the above analysis. The amplitude distribution is an even function of position within the cell and, as one goes from the lower edge of the band to the higher edge, the field form for the first pass band (diagrams C, D and E of Fig. 15) varies from a constant ($m=0$, but modulated by the higher space harmonics) to a wave with one zero within a period d ($m=1$), and the field shape for the second pass band (diagrams H, I and J of Fig. 16) varies from a wave with one zero per period ($m=1$) to one with two zeros ($m=2$).

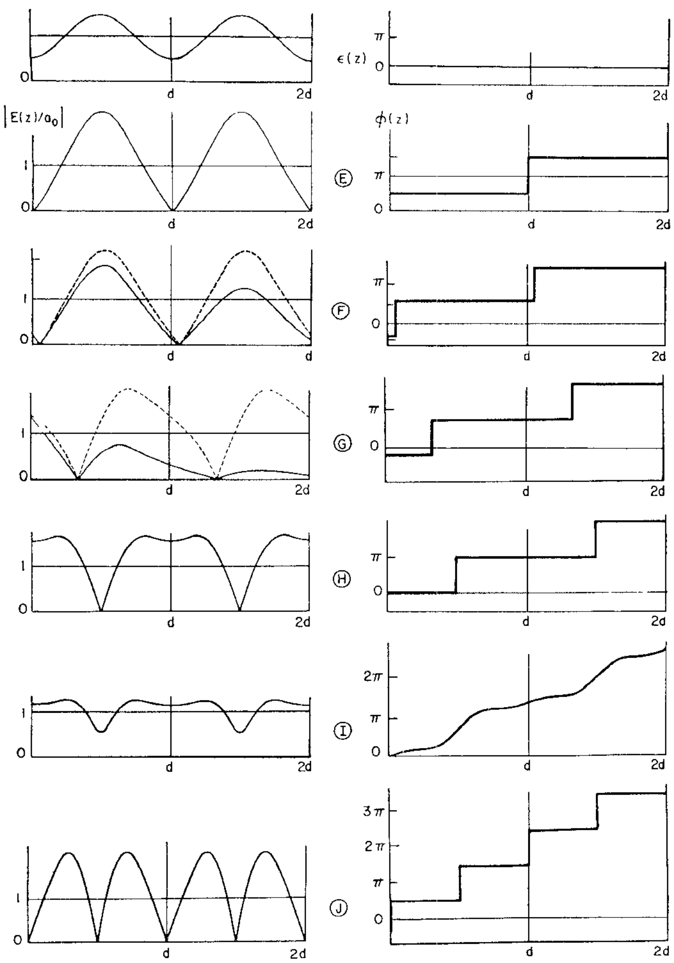


Fig. 16—Variation of the field in the second unstable (stop band) and the second stable (pass band) regions. Situations E to J are defined in Fig. 14. The $\epsilon(z)$ curves represent the actual variation in dielectric constant. Full and dashed lines refer, respectively, to $|A(z)|$ and $(A_p(z)|$ [see (58)].

Another interesting feature is that the distribution of the electric field has no direct relation to the spatial variation of the dielectric constant within the medium. In fact, it is seen in Fig. 15 for diagram D, for example, that the field has maxima wherever $\epsilon(z)$ goes through a maximum while, for diagram H of Fig. 16, the field possesses zeros at the same locations. Moreover, in the unstable regions (diagram G in Fig. 16, for example), the extrema of the field are altogether shifted in position with respect to the extrema of $\epsilon(z)$.

The behavior associated with the phase curves of Figs. 15 and 16 is a simple corollary of the changes in the field amplitude. Within the pass bands we expect phase progression with distance to occur within the cells, and indeed it does, as seen in diagrams D and I. In the stop bands and at the band edges the phase is either constant or it changes discontinuously, as expected. Although there is now no phase progression with distance in the cell, we do expect a phase shift per cell of $m\pi$ [see (22)]. The remaining diagrams are all in agreement with this requirement, with A, B and C corresponding to $m=0$, E, F, G and H corresponding to $m=1$, and J corresponding to $m=2$. The actual jumps of π are seen to occur precisely at the zeros in amplitude, indicating merely that the fields change sign as one passes a point of zero amplitude.

VI. Conclusion

The modes of propagation in an infinite medium possessing a sinusoidally stratified dielectric constant were shown to be of two different types. These are 1) modes which are associated with pass bands of the medium: these consist of an infinite number of space harmonics, each of which is in the form of a uniform plane wave that propagates at a different angle in the medium, and 2) modes which are associated with stop bands of the medium: these also consist of an infinite number of harmonics, but now all of them propagate only along the striations formed by the modulation in the medium, and they vary exponentially in a direction normal to these striations.

The dispersion curves for the infinite stratified medium are obtainable by means of a simple geometrical construction involving a stability chart customarily used for Mathieu functions. This construction was extended to account for propagation properties in certain bounded, rather than infinite, configurations. One of these structures consists of a waveguide containing a dielectric which is modulated in the longitudinal direction; it then turns out that propagation is possible at frequencies for which the unmodulated waveguide would be below cutoff.

The aspects of small modulation were treated at length for both the infinite and the bounded configurations. Simple analytical solutions for the modes and their fields were presented for the pass bands. The modes are then given essentially by the fundamental and the two closest space harmonics, while the higher space harmonics yield negligible contributions.

The field within a unit cell of the modulated medium was calculated for small and large modulation in both the pass bands and the stop bands, as well as at the band edges. As expected, the space harmonics possess amplitudes which are equal in pairs in the stop bands; the field distribution is then in the form of a damped standing wave. In the pass bands, the fields are nondecaying and every cell introduces a net phase shift which produces the propagation associated with these bands. It is interesting to note that the shape of the field in a unit cell bears no direct relationship to the variation of the dielectric constant within the cell except at very low frequencies.